Announcements

1) Job talk Thursday, time TBA

2) Midterm week after the week after spring break

Recall: goal is to triangularize a matrix by applying unitaries.

For a general matrix

Start with $A \in \mathbb{C}^{m \times n}$.

Multiply A on the left by a unitary Q_1 where Q_1 sends the first column of A to

$$\begin{bmatrix} x \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{with } |x| = \|a_1\|_2$$

if $a_1 = 1^{st}$ column of A.

Then apply

$$Q_2 = \begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & F_2 \\ 0 & \end{bmatrix}$$

where F_2 sends the second column of $Q_1 A$ (minus the first row), a vector called b, to

$$\begin{array}{l l} \text{m-1 spots} \left\{ \right. & \begin{bmatrix} y \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{array} \quad \text{with } |y| = \|b\|_2$$

Q_3 will act on $Q_2 Q_1 A$

to produce $\left.\begin{bmatrix} z \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\}$ $m-2$ spots

on the third column, etc.

After applying $Q_1, Q_2, \ldots, Q_n$

to A in succession, we

obtain an upper-triangular

matrix R.

Then

$$R = \underbrace{\left(Q_n Q_{n-1} \cdots Q_3 Q_2 Q_1\right)}_{Q^*} A$$

Since the product of unitaries is unitary,

$$\boxed{A = QR}$$

In our example from last class:

A was 3×3, but...

the first column had a zero in it, which reduced the calculation of Q, to 2 dimensions.

In general, we want to send $a_1 = 1^{st}$ column of A to $\|a_1\|_2 e_1 \in \mathbb{C}^m$.

$m=3$, the orthogonal complement to any nonzero vector is now a plane. Not as easy as a line!

But - we are reflecting about a hyperplane orthogonal to the vector $a_1 - \|a_1\|_2 e_1$.

If we call this vector v_1, then

$$P_1 = \frac{v_1 v_1^*}{\|v_1\|^2}$$

$$= \frac{v_1 v_1^*}{v_1^* v_1}$$

is the orthogonal projection onto $\text{span}(v_1)$.

The projection onto the hyperplane is then $I_m - P_1$,

and we can obtain

$$Q_1 = 2(I_m - P_1) - I_m$$

$$= I_m - 2P_1$$

Then continuing on in this fashion,

$$Q_2 = \begin{bmatrix} 1 & 0 \cdots 0 \\ \vdots & F_2 \\ 0 & \end{bmatrix}$$

where F_2 is the unitary $I_{m-1} - 2P_2$

where P_2 is the rank one projection determined by the second column of $Q_1 A$ (minus 1^{st} row)

Here $P_2 = v_2 v_2^* / v_2^* v_2$

where, if b is the second column,

$$v_2 = b - \|b\|_2 e_1 \in \boxed{\mathbb{C}^{m-1}}$$

In general,

$$\boxed{Q_j = \begin{bmatrix} I_{j-1} & 0 \\ 0 & \underbrace{I_{m-j+1} - 2P_j}_{F_j} \end{bmatrix}}$$

where $P_j = \dfrac{v_j v_j^*}{v_j^* v_j}$

Then

$$R = \underbrace{(Q_n Q_{n-1} \cdots Q_3 Q_2 Q_1)}_{Q^*} A$$

Since the product of unitaries is unitary,

$$\boxed{A = QR}$$

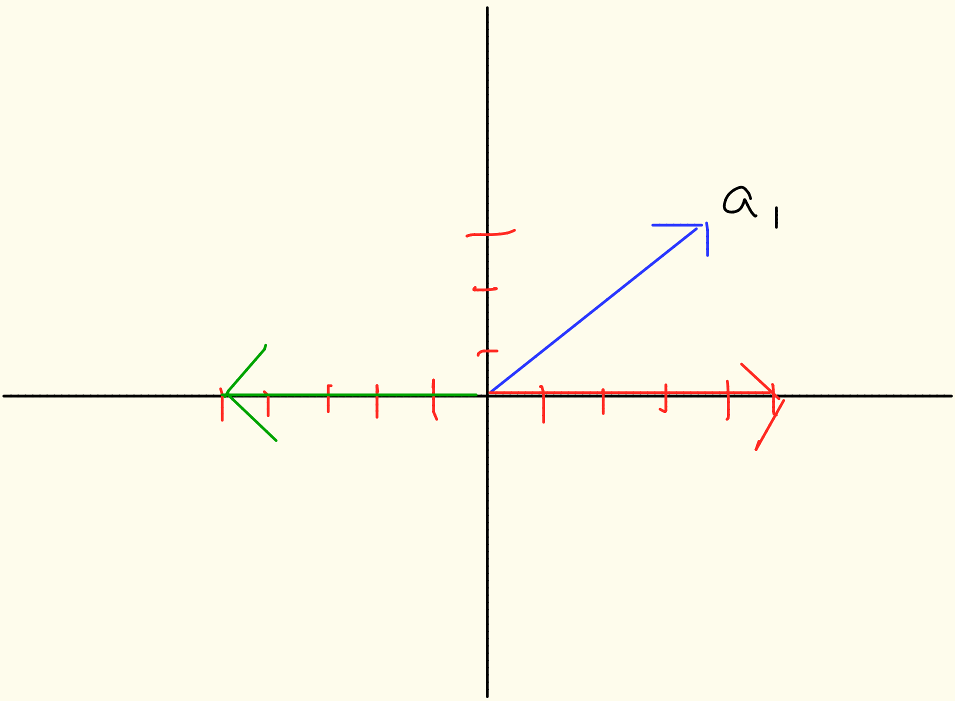
Definition: (Householder reflection)

The matrices F_i generated by the algorithm for $1 \leq i \leq n$ are called **Householder reflections**.

In general, if P is the orthogonal projection onto a one-dimensional subspace of $\mathbb{C}^m$, $I_m - 2P$ is a Householder reflection.

Choice of Signs

In our example, we had



We could have chosen the green vector instead of the red to map onto via Q_1. The "right" choice for numerical stability is the green vector! In general, we want to choose the vector "farther away" from x to prevent rounding errors.

Instead of

$$v_1 = a_1 - \|a_1\|_2 e_1,$$

we let

$$v_1 = a_1 - \text{sign}(a_{1,1}) \|a_1\|_2 e_1$$

where $a_{1,1} = 1^{st}$ coordinate of a_1.

Do this at every stage of the algorithm.